

# Rogue waves in nonlocal media

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The generation of rogue waves is investigated via a nonlocal nonlinear Schrödinger (NLS) equation. In this system, modulation instability is suppressed and is usually expected that rogue wave formation would also be limited. On the contrary, a parameter regime is identified where the instability is suppressed but nevertheless the number and amplitude of the rogue events increase, as compared to the standard NLS (which is a limit of the nonlocal system). Furthermore, the nature of these waves is investigated; while no analytical solutions are known to model these events, numerically it is shown that they differ significantly from either the rational (Peregrine) or soliton solution of the limiting NLS equation. As such, these findings may also help in rogue wave realization experimentally in these media.

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Many destructive phenomena in the oceans have been identified as the result of rogue waves, extreme waves that grow abnormally during a given sea state. This has motivated wide ranging research on rogue and extreme phenomena that spans across sciences [1–7]. Key to understanding these phenomena is their common description through the well-known nonlinear Schrödinger equation (NLS) with cubic/Kerr nonlinearity [8].

The NLS system provides a unique balance between the critical effects that govern propagation in dispersive media, namely dispersion/diffraction and nonlinearity. This balance leads to the formation of solitons, which are characterized by their stability and robustness as they maintain their shape and velocity even when they interact. Rogue waves, on the other hand, are observed to appear from nowhere and are short-lived. They are often modeled by the so-called Peregrine soliton [9, 10], which is a special type of solitary wave formed on top of a continuous wave (cw) background and in contrast to other soliton solutions of the NLS equation it is written in terms of rational functions with the property of being localized in both time and space. These properties make these solutions useful to describe such events [11, 12].

The specific conditions that cause their formation is still a subject of enormous interest; it is generally recognized that modulation instability (MI) is among the most important mechanisms which lead to rogue wave excitation [13–18]. MI is the nonlinear mechanism of the self-wave interactions, called the Benjamin-Feir instability [19] in water wave physics. In nonlinear optics, it is considered as a basic process that classifies the qualitative behavior of modulated waves [20]. Rogue waves, as a result of an MI process, can be identified as high-contrast peaks of random intensity and are the result of the unstable growth of weak wave modulations. Mathematically, MI is a fundamental property of many nonlinear dispersive systems and is a well documented and understood phenomenon [21].

The NLS equation provides important information

about rogue phenomena. However, for several physically relevant contexts the standard focusing NLS equation turns out to be an oversimplified description as it cannot model, for example, gain and loss which are inevitable in any physical system [22]. Hence, in order to model important classes of physical systems in a relevant way, it is necessary to go beyond the standard NLS description. There are, for example, important systems that display nonlocal nonlinear mechanisms. Such media include, nematic liquid crystals [23, 24], thermal nonlinear optical media [25, 26] and plasmas [27, 28].

The effect of the nonlocality on the NLS equation is rather profound. The integrable nature of the equation is generally lost and while soliton solutions may also be found they lack a free parameter linking their amplitude to their velocity. In terms, of rational (rogue type) solutions none are known, to our knowledge. In terms of the MI properties in the model we investigate, the cw solutions are always unstable with the nonlocality suppressing the instability (although it does not eliminate the effect) [29]. In fact, the effect is so strong that it has been proven that the nonlocality eliminates collapse in all physical dimensions [30]. These observations suggest that the nonlocality has a stabilizing effect. As such, it might be expected that rogue wave phenomena should be more scarce and more prominent in the weakly nonlocal regime where the system is closer to the standard NLS equation. We find, here, that the nonlocality does not always suppress the number and size of the rogue events.

The normalized system that governs propagation in nonlocal media reads [24, 31]

$$i\frac{\partial u}{\partial z} + d\frac{\partial^2 u}{\partial x^2} + 2g\theta u = 0 \quad (1a)$$

$$\nu\frac{\partial^2 \theta}{\partial x^2} - 2q\theta = -2|u|^2 \quad (1b)$$

Depending on the physical situation the system and its coefficients correspond to different physical quantities. For example, in the context of nematic liquid crystals,

$u$  is the complex valued, slowly varying envelope of the optical electric field and  $\theta$  is the optically induced deviation of the director angle. Diffraction is represented by  $d$  and nonlinear coupling by  $g$ . The effect of nonlocality  $\nu$  measures the strength of the response of the nematic in space, with a highly nonlocal response when  $\nu$  is large. The parameter  $q$  is related to the square of the applied static field which pre-tilts the nematic dielectric [32]. In this context,  $d, g, q$  are  $O(1)$  while  $\nu$  is  $O(10^2)$  [24, 32].

In order to investigate the stability properties of system (1) consider its cw wave solution

$$u(z) = u_0 e^{2ig\theta_0 z}, \theta_0 = \frac{1}{q} u_0^2$$

where  $u_0$  is a real constant. Consider a small perturbation to this cw solution

$$u(x, z) = [u_0 + u_1(x, z)] e^{2ig\theta_0 z}$$

which is assumed to behave as  $\exp[i(kx - \omega z)]$  provided the dispersion relation:

$$\omega^2 = \frac{dk^2 (d\nu k^4 + 2dqk^2 - 8gu_0^2)}{\nu k^2 + 2q} \quad (2)$$

It is clear that when  $dg > 0$  the system is unstable (and is termed focusing) whereas when  $dg < 0$  the system is stable (and is termed defocusing). Also, when  $\nu = 0$  the equation reduces to the dispersion relation of the relative NLS equation, which has the same stability criteria. From this dispersion relation we can identify three critical values that characterize the instability, namely the maximum growth rate,  $\text{Im}\{\omega_{\max}\}$  and its location,  $k_{\max}$ , and the width of the instability region,  $k_c$ . The first,  $\text{Im}\{\omega_{\max}\}$ , is a measure of the propagation distance needed for the instability to occur (the higher its value the faster the instability occurs) and the last,  $k_c$ , defines the range of possible wavenumbers that can deem the system unstable (the higher its value the more unstable the system as more cw's can cause an unstable propagation). By differentiating Eq. (2), with respect to  $k$ , we find that  $k_{\max}$  is the solution of the algebraic equation

$$d(\nu k^3 + 2qk)^2 - 8gqu_0^2 = 0$$

while  $k_c$  satisfies

$$d\nu k^4 + 2dqk^2 - 8gu_0^2 = 0$$

Both equations can be solved in closed form (they are bi-quadratics) to give the relative dependance of  $\text{Im}\{\omega_{\max}\}$  and  $k_c$  with the nonlocality  $\nu$ . We illustrate this in Fig. 1. Hereafter we fix  $d = 1/2$  and  $g = q = u_0 = 1$ .

This figure confirms the findings of Ref. [29] that the nonlocality has a stabilizing effect in the system. Indeed, both critical values that characterize the instability,  $\text{Im}\{\omega_{\max}\}$  and  $k_c$ , decrease as  $\nu$  increases. This means

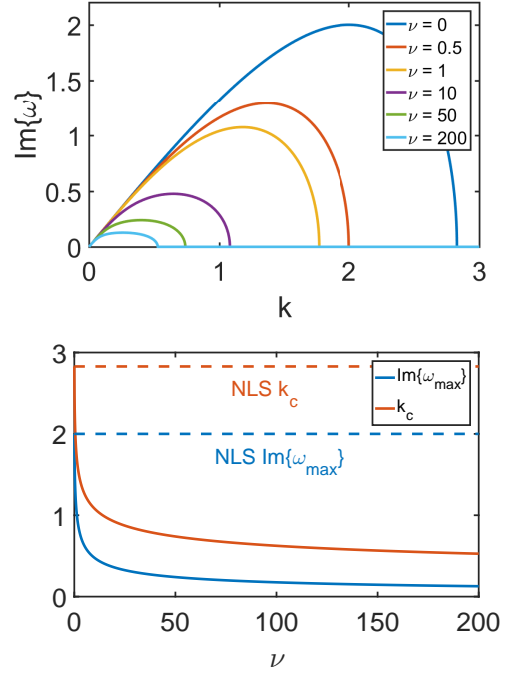


Figure 1: (Color Online) Top: Growth rates for different values of the nonlocal parameter  $\nu$ . Bottom: The change of critical values  $\text{Im}\{\omega_{\max}\}$  and  $k_c$  with the nonlocality  $\nu$ .

that the effect will need more distance to be exhibited (and if  $\nu$  is large enough this distance can be larger than the experimental scales) and that a smaller range of wave numbers will cause an instability. Notice, again, that while both values decrease, the effect, in the focusing case, is always present, just suppressed. The limiting NLS system is, by these values, significantly more unstable.

To see how these observations affect the generation of rogue waves, we integrate numerically Eq. (1) using a pseudospectral method in space and exponential Runge-Kutta for the evolution [33] in a computational domain  $x \in [-100, 100]$ ,  $z \in [0, 20]$ . An appropriate initial condition would be a wide gaussian of the form

$$u(x, 0) = v(x, 0) = e^{-x^2/2\sigma^2}, \sigma = 30$$

perturbed with additional 10% random noise. A wide gaussian with randomness added is a prototype of a set of broad/randomly generated states which can potentially excite more than one wave numbers as it can be regarded as a Fourier series of different cw's of different  $k$ 's. This is particularly important here as a single cw initial condition may not cause any growth due to the decrease of  $k_c$  with  $\nu$ . For each value of the parameter  $\nu$  we perform  $10^5$  trials. In each trial we measure the highest wave amplitude and introduce the quantity

$$\tilde{u}(x, z) = \frac{u(x, z)}{\max\{u(x, 0)\}}$$

which measures the relative growth in amplitude from an initial state. Here we consider a rogue event as one in which  $\tilde{u}(x, z)$  at some value of  $z$  is at least three times its maximum initial value. In Fig. 2 we depict the change in rogue events for some values of  $\nu$ .

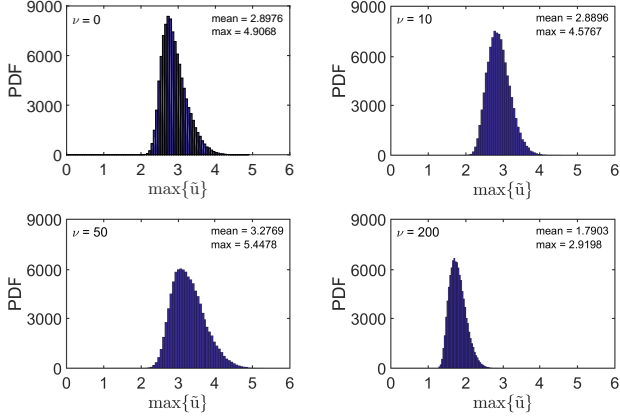


Figure 2: (Color Online) Probability density functions of the maximum value or  $\max\{\tilde{u}\}$  for different values of the nonlocal parameter  $\nu$ .

These PDFs indicate that there is a relationship between the occurrence of rogue events and nonlocality. Indeed, starting with  $\nu = 10$  the mean of the PDF is comparable to that of the regular NLS ( $\nu = 0$ ). With  $\nu = 50$  there is a definite shift towards the right indicating that rogue events have increased in both numbers and severity (amplitude). Finally, for  $\nu = 200$  there is a sharp decrease of events and their amplitudes. This indicates that there is a nontrivial dependence between the nonlocality and the occurrence of rogue events. The expectation that nonlocality stabilizes the system and thus suppresses extreme phenomena does not hold.

To further investigate the dependence of rogue events with  $\nu$ , we perform the same analysis for a wide range of the parameter. In Fig. 3 we depict the change of the mean value in the PDFs for the maximum values of  $\tilde{u}$  with  $\nu$  as well as the change of the top 10% of the highest valued events.

Based on this figure, there are three different regions of interest. At first, when  $0 \leq \nu \lesssim 10$ , there is a sharp drop from the NLS case ( $\nu = 0$ ) to about  $\nu = 1$  and then both curves increase with  $\nu$ , but still remain below the NLS limit. Of particular interest is the transition from  $\nu = 0$ . While we have taken points of order  $10^{-2}$  in  $\nu$  (in this region) there is a sharp drop, a boundary layer type change, to a local minimum after which both the mean and max curves increase. Next, in the region  $10 \lesssim \nu \lesssim 110$ , the curves remain well above the NLS limits which translates into the system producing more numerous and more extreme events. Recall, again that for these values the system exhibits very weak growth rates and has a very narrow instability band. Finally, for

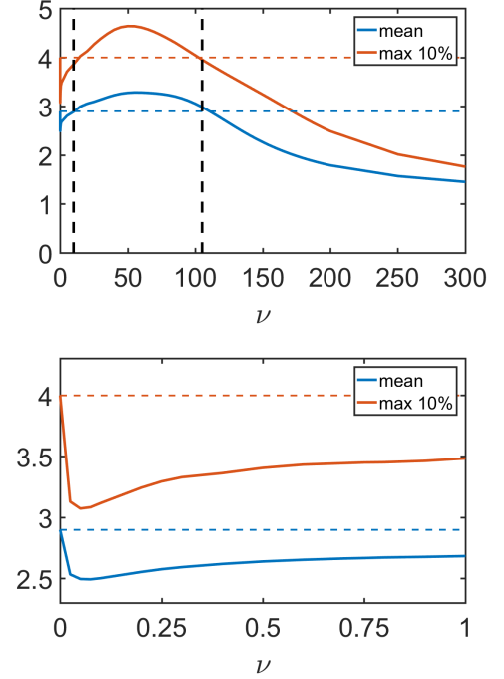


Figure 3: (Color Online) Top: The mean value of the PDFs and the mean value of the max 10% events with the nonlocal parameter  $\nu$ . The horizontal dashed lines indicate the relative values for  $\nu = 0$  and the vertical dashed lines the values of  $\nu$  for which these values surpass the NLS system. Bottom: A zoom in around  $\nu = 0$ .

$\nu > 110$  the expected behavior is observed namely both curves slowly decay as the nonlocal parameter increases.

Next, considering the region where rogue wave are maximized, we now turn our attention to the nature of these waves. Indeed, an important aspect of rogue wave formation is the type or shape of the event, frequently modeled by the so-called Peregrine soliton, a rational solution which reads for Eqs. (1) (with  $\nu = 0$ )

$$u_P(x, z) = u_0 \left[ 1 - \frac{4dq^2 + i(16dgqu_0^2)z}{dq^2 + (4gqu_0^2)x^2 + (16dg^2u_0^4)z^2} \right] e^{2igu_0^2z/q}$$

while the single soliton solution is

$$u_s(x, z) = u_0 \text{sech}(u_0 \sqrt{g/dq} x) e^{iu_0^2 g z / q}$$

It is counter intuitive (and verified below) to believe that either would be a good candidate to approximate rogue waves in this context as they lack the dependence on the nonlocal parameter  $\nu$ . Furthermore, the soliton solution of Eqs. (1) is [34]

$$u(x, z) = \frac{3q}{2} \sqrt{\frac{d}{g\nu}} \text{sech}^2(\sqrt{q/2\nu} x) e^{2idq/\nu z}$$

which while it obviously depends on  $\nu$ , it has fixed amplitude (much like  $\chi^{(2)}$  materials [35, 36]) which decays

with  $\nu$ . As such, this solution is again not an appropriate candidate to model extreme events (higher nonlocality results in smaller soliton amplitudes). In fact, solutions with a free parameter for this system have been found but only in the defocusing case and under a small amplitude approximation technique [37]. To illustrate we compare all these solutions to an arbitrary rogue event in Fig. 4.

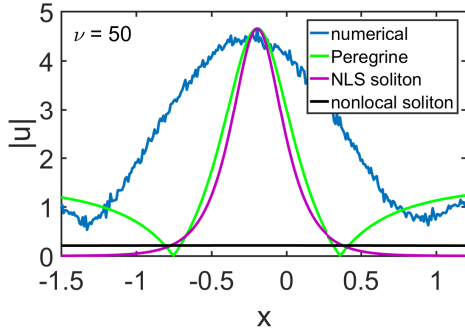


Figure 4: (Color Online) Comparison of a (randomly chosen) rogue event of the nonlocal equation with the known soliton and rational solutions.

Clearly the two solutions of the regular NLS system ( $\nu = 0$ ) are too narrow to fit the event, while the decaying soliton of the nonlocal system is of the order 0.2 and appears as a straight (black) line due to its magnitude. To further investigate the matter, in Fig. 5, we zoom in around a rogue event for different values of the parameter  $\nu$  and fit a rational solution around it.

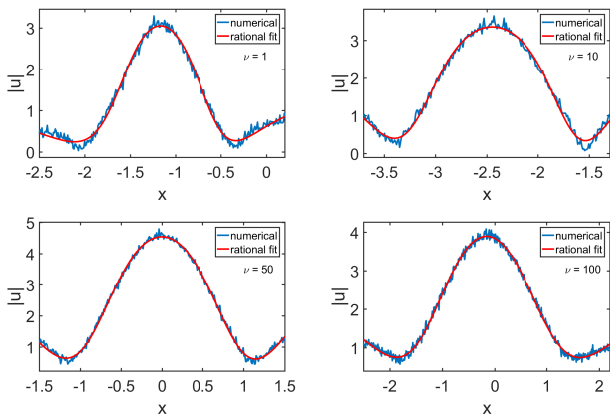


Figure 5: (Color Online) A zoom in around a rogue event for the different values of the nonlocal parameter  $\nu$ . A fourth order rational solution has been fitted (red line) in all cases.

The best fit is given by the ratio of two fourth order polynomials in  $x$ . We notice that the fits become increasingly better as  $\nu$  increases indicating the profound difference with the integrable system. This is consistent with the soliton solutions. Indeed, the sech-type soliton

of the NLS is replaced by the sech<sup>2</sup>-solution of the nonlocal system. This is not the first time that more general (and commonly not known to be integrable) systems give rogue events whose nature differs from that of the typical rational Peregrine soliton. A similar situation was recently observed in deep water waves [38].

To conclude, we have studied rogue wave formation in nonlocal media using a physically important nonlocal NLS system. For these systems, MI is suppressed in both the strength of growth rates and size of instability band. Common belief suggests that this would also result in the appearance of fewer and smaller, in amplitude, events. Contrary to that we found that for a wide range of values of the nonlocal parameter, the system may produce significantly more events in both size and numbers. The only known soliton solution of the system is not suited to describe these events which also differ from their Kerr type counterparts in that they are well approximated by fourth order rational solutions.

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- [1] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali, *Nature Lett.* **450**, 1054 (2007).
  - [2] A. Chabchoub, N. P. Hoffmann, and N. Akhmediev, *Phys. Rev. Lett.* **106**, 204502 (2011).
  - [3] M. Shats, H. Punzmann, and H. Xia, *Phys. Rev. Lett.* **104**, 104503 (2010).
  - [4] A. N. Pisarchik, R. Jaimes-Reátegui, R. Sevilla-Escoboza, G. Huerta-Cuellar, and M. Taki, *Phys. Rev. Lett.* **107**, 274101 (2011).
  - [5] Y. Zhen-Ya, *Commun. Theor. Phys.* **54**, 947 (2010).
  - [6] Y. V. Bludov, V. V. Konotop, and N. Akhmediev, *Phys. Rev. A* **80**, 033610 (2009).
  - [7] A. Zaviyalov, O. Egorov, R. Iliev, and F. Lederer, *Phys. Rev. A* **85**, 013828 (2012).
  - [8] M. J. Ablowitz, *Nonlinear Dispersive Waves* (Cambridge University Press, 2011).
  - [9] D. H. Peregrine, *J. Austral. Math. Soc. Ser. B* **25**, 16 (1983).
  - [10] N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, *Theoret. and Math. Phys.* **72**, 809 (1987).
  - [11] N. Akhmediev, J. M. Soto Crespo, and A. Ankiewicz, *Phys. Lett. A* **373**, 2137 (2009).
  - [12] F. Baronio, A. Degasperis, M. Conforti, and S. Wabnitz, *Phys. Rev. Lett.* **109**, 044102 (2012).
  - [13] V. E. Zakharov, A. I. Dyachenko, and A. O. Prokofiev, *Eur. J. Mech. B Fluids* **25**, 677 (2006).
  - [14] V. E. Zakharov and A. A. Gelash, *Phys. Rev. Lett.* **111**, 054101 (2013).
  - [15] N. Akhmediev, J. M. Soto-Crespo, and A. Ankiewicz, *Phys. Rev. A* **80**, 043818 (2009).
  - [16] M. Onorato, S. Residori, U. Bortolozzo, A. Montina, and F. T. Arecchi, *Phys. Reports* **528**, 47 (2013).
  - [17] F. Baronio, S. Chen, P. Grelu, S. Wabnitz, and M. Conforti, *Phys. Rev. A* **91**, 033804 (2015).

- [18] M. Erkintalo, K. Hammani, B. Kibler, C. Finot, N. Akhmediev, J. M. Dudley, and G. Genty, *Phys. Rev. Lett.* **107**, 253901 (2011).
- [19] T. B. Benjamin and J. E. Feir, *J. Fluid Mech.* **27**, 417 (1967).
- [20] V. E. Zakharov and L. A. Ostrovsky, *Physica D* **238**, 540 (2009).
- [21] J. M. Dudley, F. Dias, M. Erkintalo, and G. Genty, *Nature Photonics* **8**, 755 (2014).
- [22] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic Press, 2013).
- [23] C. Conti, M. Peccianti, and G. Assanto, *Phys. Rev. Lett.* **91**, 073901 (2003).
- [24] G. Assanto, *Nematicons: Spatial Optical Solitons in Nematic Liquid Crystals* (Wiley-Blackwell, 2012).
- [25] C. Rotschild, O. Cohen, O. Manela, M. Segev, and T. Carmon, *Phys. Rev. Lett.* **95**, 213904 (2005).
- [26] W. Krolikowski, O. Bang, N. I. Nikolov, D. Neshev, J. Wyller, J. J. Rasmussen, and D. Edmundson, *J. Opt. B: Quantum Semiclass. Opt.* **6**, S288 (2004).
- [27] A. G. Litvak, V. A. Mironov, G. M. Fraiman, and A. D. Yunakovskii, *Sov. J. Plasma Phys.* **1**, 60 (1975).
- [28] A. I. Yakimenko, Y. A. Zaliznyak, and Y. S. Kivshar, *Phys. Rev. E* **71**, 065603(R) (2005).
- [29] W. Krolikowski, O. Bang, J. J. Rasmussen, and J. Wyller, *Phys. Rev. E* **64**, 016612 (2001).
- [30] O. Bang, W. Krolikowski, J. Wyller, and J. J. Rasmussen, *Phys. Rev. E* **66**, 046619 (2002).
- [31] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, 2003).
- [32] M. Peccianti and G. Assanto, *Phys. Reports* **516**, 147 (2012).
- [33] A. Kassam and L. N. Trefethen, *SIAM J. Sci. Comput.* **26**, 1214 (2005).
- [34] J. M. L. MacNeil, N. F. Smyth, and G. Assanto, *Physica D* **284**, 1 (2014).
- [35] Y. N. Karamzin and A. P. Sukhorukov, *JETP Lett.* **20**, 339 (1974).
- [36] A. V. Buryak, P. D. Trapani, D. V. Skryabin, and S. Trillo, *Phys. Reports* **370**, 63 (2002).
- [37] T. P. Horikis, *J. Phys. A: Math. Theor.* **48**, 02FT01 (2015).
- [38] M. J. Ablowitz and T. P. Horikis, *Phys. Fluids* **27**, 012107 (2015).